

# Wave Probe of Hořava-Lifshitz Gravity

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Particle probe considered in the Kehagias - Sfetsos black hole of Hořava-Lifshitz gravity by Gwak and Lee (JCAP, 1009: 031, 2010) is reconsidered within the framework of quantum mechanics. The timelike naked singularity that develops when  $\omega M^2 < 1/2$ , is probed with quantum fields obeying the Klein-Gordon and Chandrasekhar-Dirac equations. Quantum field probe of the naked singularity has revealed that the spatial part of the wave operators of the Klein-Gordon and Chandrasekhar-Dirac equations are essentially self-adjoint and thus, the naked singularities in the Kehagias - Sfetsos spacetime becomes quantum mechanically nonsingular.

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## I. INTRODUCTION

One of the most challenging problem of the theoretical physics is, how to merge the physics occurring at small scales (quantum level) to those at large scales (classical general relativity). Resolution of this problem is extremely important, because at small scales the classical general relativity breaks down and the description of gravitational interaction becomes not possible. The efforts toward constructing a consistent quantum theory of gravity has encountered with serious obstacles. One of these obstacles is that the Einstein's theory of classical general relativity (perturbatively) is not a renormalizable theory and thus, the conventional quantization techniques are not applicable.

However, an alternative modified theories are developed to serve for the resolution of this problem. String [1, 2] and loop quantum gravity [3] theories are developed for dealing with the problems at small scales. It has been shown in string theory that some timelike singularities are resolved.

In recent years, there has been a growing interest to another alternative theory within the context of quantum gravity, —the Hořava - Lifshitz (HL) theory of gravity [4, 5]. The HL theory incorporates with an anisotropic scaling of space and time. As a consequence of this scaling, while the Lorentz invariance is broken at high energies (short distances, UV regime), the Lorentz invariance is recovered at low energies (IR regime). The HL gravity theory "could therefore serve as a UV completion of Einstein's general relativity" [5]. Theory is also called in the literature as the power-counting renormalizable theory. Currently, there are several versions of HL theory that can be classified whether or not the detailed balance and projectability conditions are imposed.

In the literature, there are variety of studies related to the HL gravity. The developments in this theory are collected and presented in a recent progress report, prepared by Wang [6]. Among the others, the spherically symmetric solutions having characteristics analog to the Schwarzschild solution has attracted considerable amount of interest. In particular, the solution found by Kehagias and Sfetsos (KS) [7, 8], which describes static, spherically symmetric black hole solution in the limiting case of  $(3 + 1)$ —dimensional HL gravity. Whether in classical general relativity or in modified theories, solutions admitting black holes are always more fascinating and, as a result, attracting more attention. For example, the underlying physics of the KS black hole solutions are extensively studied in terms of geodesics (particle motion) [9–15].

However, solutions admitting naked singularities are always undervalued both in classical general relativity and in modified theories. One reason that may be linked to this view is the violation of the Penrose's weak cosmic censorship hypothesis (CCH). According to this hypothesis, all singularities in physically realistic spacetimes are hidden by the horizons of black holes, which preserves the deterministic nature of the classical general relativity. However, many solutions have been found to the Einstein's equations that may exhibit naked singularities. The formation of the naked singularity in the KS solution can be given as an example. And, the purpose of this paper is to investigate this naked singularity within the framework of quantum mechanics.

Spacetime singularities are predicted by Einstein's theory of general relativity and described as the *geodesics incompleteness* with respect to the point particle probe. Spacetime is geodesically incomplete, if it contains at least one geodesic that is inextendible. In other words, at the singularities, all the laws of physics are broken down, and that

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is why, all these alternative theories are emerging to resolve this challenging problem. Because, before reaching the singularity, we are in a microscopic region of the space that instead of the laws of classical general relativity, the laws of quantum gravity is expected to be replaced. Hence, any attempt for investigating singularities in conjunction with quantum mechanics should be considered as an important step in the right direction.

In this study, the KS naked singularity spacetimes representing spherically symmetric static vacuum solution in the HL gravity will be investigated within the framework of quantum mechanics. The KS metric incorporates with two parameters, the gravitational mass parameter  $M$  and the Hořava parameter  $\omega$ , which represents the influence of the quantum effects. These two parameters determines the physical character of the KS spacetime. If the product  $\omega M^2 \geq \frac{1}{2}$ , the KS metric possesses a black hole solution with two horizons. Thus, the curvature singularity at  $r = 0$  is covered by these horizons and preserves the CCH. However, if the product  $\omega M^2 < \frac{1}{2}$ , there are no horizons and the curvature singularity at  $r = 0$  becomes visible to asymptotic observers, which is called the naked singularities. The observational constraints on the value of  $\omega$  presented in [12, 16, 17], do not exclude the existence of the KS naked singularities. Hence, in the light of these observational facts, it is very important to focus on the naked singularities in the KS spacetimes. It is shown in [18] that the optical signatures of the KS naked singularities is different from the signatures of the standard black holes in classical general relativity. Furthermore, circular geodesics in the KS naked singularity spacetimes is studied in [19] and compared with the counterparts in classical general relativity.

In this paper, we focus on the quantum nature of the KS naked singularity. We investigate whether this classically singular spacetime remains quantum mechanically singular or not. In our analysis, quantum particles (fields) obeying the Klein-Gordon and the Dirac equations will be sent to the KS naked singularity. Thus, our analysis will be based on a wave probe, which leads to the notion of *quantum singularity*. In doing this, the work of Wald [20], which was developed by Horowitz and Marolf (HM) [21] for static spacetimes will be used. The criterion of HM incorporates with quantum field theory in curved spacetime. Hence, the analysis is based on the motion of quantum particles (fields) in a classical curved background.

The paper is organized as follows. In section II, we give the brief review of the KS spacetime and the description of the KS metric in a Newman-Penrose formalism. In section III, the HM criterion is briefly explained. The naked singularity in the KS spacetime is analysed by probing the singularity with quantum fields obeying the Klein-Gordon and Dirac equations. The paper ends with a conclusion and discussion in section IV.

## II. REVIEW OF THE KS SPACETIME

The 3 + 1-dimensional action describing the Hořava - Lifshitz gravity is given in [5] as,

$$S = \int dt d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \frac{\kappa^2}{2\omega^4} C_{ij} C^{ij} + \frac{\kappa^2 \mu}{2} \left( \frac{\epsilon^{ijk}}{\omega^2} R_{il}^{(3)} \nabla_j R_k^{(3)l} - \frac{R_{ij}^{(3)} R^{(3)ij}}{4} \right) \right. \\ \left. + \frac{\kappa^2 \mu}{8(1-3\lambda)} \left( \frac{1-4\lambda}{4} (R^{(3)})^2 + \Lambda_W R^{(3)} - 3\Lambda_W^2 \right) + \mu^4 R^{(3)} \right\}, \quad (1)$$

in which

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (2)$$

is the second fundamental form and

$$C^{ij} = \epsilon^{ikl} \nabla_k \left( R_l^{(3)j} - \frac{1}{4} R^{(3)} \delta_l^j \right), \quad (3)$$

is the Cotton tensor,  $\kappa, \lambda$  and  $\omega$  are dimensionless coupling constants. On the other hand,  $\mu$  and  $\Lambda_W$  are dimensionful of mass  $[\mu] = 1$  and  $[\Lambda_W] = 2$ . The corresponding metric is given by

$$ds^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (4)$$

where  $g_{ij}, N^i$  and  $N$  are the dynamical fields of scaling mass dimensions 0, 2, 0, respectively.

The above Lagrangian has been considered by Kehagias and Sfetsos in the limiting case of  $\Lambda_W \rightarrow 0$  and  $\lambda = 1$ , which leads, perhaps, one of the important solution obtained so far within the context of Hořava - Lifshitz gravity. The obtained solution is spherically symmetric, which describes asymptotically flat black hole metric. This metric

is important, because, it constitutes the analog of Schwarzschild solution of classical general relativity. The metric obtained by KS is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (5)$$

in which

$$N^2 = f(r) = 1 + \omega^2 r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}, \quad (6)$$

where  $M$  is an integration constant with dimension  $[M] = -1$  and  $\omega = 16\mu^2/\kappa^2$ . The obtained metric possesses black hole solution with two horizons located at

$$r_{\pm} = M \left( 1 \pm \sqrt{1 - \frac{1}{2\omega M^2}} \right), \quad (7)$$

provided that  $\omega M^2 \geq \frac{1}{2}$ . The Ricci scalar diverges as  $1/r^{3/2}$ , indicating true curvature singularity at  $r = 0$ , which is covered by horizons. The metric (5), has interesting properties that for large  $r$  in fixed  $\omega$  or large  $\omega$  in fixed  $r$ , possesses usual Schwarzschild black hole behavior. This is the case whenever  $r \gg (M/\omega)^{1/3}$ , which allows the metric function (6) to be written in the following form,

$$f(r) \approx 1 - \frac{2M}{r} + \mathcal{O}(r^{-4}). \quad (8)$$

The classical properties of this particular case has been analysed in [15], by investigating particle geodesics. It is shown in the analysis that the KS black hole solution is flatter compared with Schwarzschild black hole in and around horizon, and as a result of this effect, the gravitation becomes weaker near the center of the region. The effective potential of the KS black hole, as in the case of Reissner-Nordström black, has a repulsive character. The overall effect of these properties on the particle motion is that no particle fall to center like Schwarzschild black hole, but particles are scattered to infinity or trapped in a periodical orbits.

#### A. The Description of the KS solution in a Newman-Penrose Formalism

The KS metric is investigated with the Newman-Penrose (NP) formalism, in order to clarify the contribution of the parameter  $\omega$ . The set of proper null tetrads  $1 - forms$  is given by

$$\begin{aligned} l &= dt - \frac{dr}{f(r)}, \\ n &= \frac{1}{2} (f(r)dt + dr), \\ m &= -\frac{r}{\sqrt{2}} (d\theta + i \sin \theta d\varphi), \\ \bar{m} &= -\frac{r}{\sqrt{2}} (d\theta - i \sin \theta d\varphi) \end{aligned} \quad (9)$$

The non-zero spin coefficients in these tetrads are

$$\beta = -\alpha = \frac{\cot \theta}{2\sqrt{2}r}, \quad \rho = -\frac{1}{r}, \quad \mu = -\frac{f(r)}{2r}, \quad \gamma = \frac{1}{4} \frac{df(r)}{dr}. \quad (10)$$

The nonzero Weyl and the Ricci scalars are

$$\Psi_2 = -\frac{M}{r^3} \left( 1 + \frac{4M}{\omega r^3} \right)^{-1/2}, \quad (11)$$

$$\phi_{11} = \frac{9M^2}{2r^6\omega} \left( 1 + \frac{4M}{\omega r^3} \right)^{-3/2}, \quad (12)$$

$$\Lambda = -\frac{\omega}{2} + \frac{\omega}{2} \left(1 + \frac{4M}{\omega r^3}\right)^{-1/2} + \frac{M}{r^3} \left(1 + \frac{4M}{\omega r^3}\right)^{-3/2} + \frac{5M^2}{2r^6\omega} \left(1 + \frac{4M}{\omega r^3}\right)^{-3/2}. \quad (13)$$

The spacetime character is Petrov *type*  $-D$ , since, the only nonzero Weyl scalar is  $\Psi_2$ . The parameter  $\omega$  represents the contribution of HL gravity. In the large limit of  $\omega \gg 1$ , the Ricci scalars  $\phi_{11}$  and  $\Lambda$  vanishes, and the Weyl scalar  $\Psi_2$  remains the only nonzero component with a value of  $\Psi_2 \approx -\frac{M}{r^3}$ , as in the case of Schwarzschild black hole.

### III. QUANTUM PROBE OF THE KS NAKED SINGULARITY

As explained with justifications in the introduction, the main purpose of this paper is to analyse the naked singularity of the KS spacetime with quantum particles/fields. To serve to purpose, the criterion developed by HM will be used in this study. According to this criterion; the classically singular spacetime remains quantum mechanically singular, if the spatial part of the wave operator is not essentially self-adjoint. If this is the case, then, the future time-evolution is not uniquely determined and hence, the corresponding spacetime is regarded as quantum mechanically singular or quantum singular. Thus, the HM criterion requires a unique time-evolution in order to say that the corresponding spacetime is quantum mechanically regular or quantum regular.

The general mathematical formalism of this criterion is given in detail in [22–24]. At this stage, we prefer to give the main theme of the HM criterion. Let us consider a relativistic scalar particle/field with mass  $\tilde{m}$  satisfying the Klein-Gordon equation. The key point is to split the spatial and time part of the Klein-Gordon equation and write it in the form of

$$\frac{\partial^2 \psi}{\partial t^2} = -A\psi, \quad (14)$$

where  $A$  is the spatial wave operator. Note that, this operator is a symmetric and positive operator on the Hilbert space  $\mathcal{H}$  [21]. According to the HM criterion, the singular character of the spacetime with respect to quantum probe is characterized by investigating whether the spatial part of the wave operator  $A$  has a unique self-adjoint extensions (i.e. essentially self-adjoint) in the entire space or not. If the extension is unique, it is said that the space is quantum mechanically regular. In order to make this point more clear, consider the Klein-Gordon equation for a free particle that satisfies

$$i\frac{d\psi}{dt} = \sqrt{A_E}\psi, \quad (15)$$

whose solution is

$$\psi(t) = e^{-it\sqrt{A_E}}\psi(0), \quad (16)$$

in which  $A_E$  denotes the extension of the wave operator  $A$ . If  $A$  has not a unique self-adjoint extensions, then the future time evolution of the wave function (16) is ambiguous. And, HM criterion defines the spacetime as quantum mechanically singular. But, if the wave operator  $A$  has a unique self-adjoint extension, then the future time evolution of the quantum particle described by (16) is uniquely determined by the initial conditions and the criterion of HM, defines this spacetime as quantum mechanically regular.

In this study, the naked singularity of the KS spacetime will be probed with two different quantum particles: *spin*  $-0$  scalar particles obeying the Klein-Gordon equation and *spin*  $-1/2$  particles obeying the Dirac equation.

#### A. The Klein-Gordon fields

The massive Klein-Gordon equation in general is given by,

$$\left(\frac{1}{\sqrt{g}}\partial_\mu[\sqrt{g}g^{\mu\nu}\partial_\nu] - \tilde{m}^2\right)\psi = 0, \quad (17)$$

in which  $\tilde{m}$  is the mass of the scalar particle. The Klein-Gordon equation is written for the metric (5) and after separating time and spatial parts, we have

$$\begin{aligned} \frac{\partial^2 \psi}{\partial t^2} = & f^2(r) \frac{\partial^2 \psi}{\partial r^2} + \frac{f(r)}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{f(r)}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{f(r) \cot \theta}{r^2} \frac{\partial \psi}{\partial \theta} + f(r) \left( \frac{2f(r)}{r} + f'(r) \right) \frac{\partial \psi}{\partial r} \\ & - f(r) \tilde{m}^2 \psi. \end{aligned} \quad (18)$$

When we compare equations (18) and (14), the spatial part of the wave operator is written as

$$A = -f^2(r) \frac{\partial^2}{\partial r^2} - \frac{f(r)}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{f(r)}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} - \frac{f(r) \cot \theta}{r^2} \frac{\partial}{\partial \theta} - f(r) \left( \frac{2f(r)}{r} + f'(r) \right) \frac{\partial}{\partial r} + f(r) \tilde{m}^2$$

The next step is to test the spatial part of the wave operator  $A$  for essential self-adjointness. We do this by applying the standard method that involves the concept of deficiency indices discovered by Weyl [25] and generalized by von Neumann [26]. The determination of the deficiency indices  $(n_+, n_-)$  of the operator  $A$ , is reduced to count the number of solutions to equation

$$(A^* \pm i) \psi = 0, \quad (19)$$

that belong to the Hilbert space  $\mathcal{H}$ . If there are no square integrable  $(L^2(0, \infty))$  solutions (i.e.,  $n_+ = n_- = 0$ ) in the entire space, the operator  $A$  possesses a unique self-adjoint extension and it is called essentially self-adjoint. Consequently, the method to find a sufficient condition for the operator  $A$  to be essentially self-adjoint is to investigate the solutions satisfying equation (19) that do not belong to the Hilbert space  $\mathcal{H}$ .

Applying separation of variables to equation (19), in the form of  $\psi = R(r) Y_l^m(\theta, \varphi)$ , yields the following radial equation for  $R(r)$ :

$$R'' + \frac{(r^2 f')}{f r^2} R' - \left[ \frac{l(l+1)}{f r^2} + \tilde{m}^2 \pm \frac{i}{f^2} \right] R = 0, \quad (20)$$

in which prime denotes the derivative with respect to  $r$ .

The square integrability of the solutions of (20) for each sign  $\pm$  is checked by calculating the squared norm, in which the function space on each  $t = \text{constant}$  hypersurface  $\Sigma_t$  is defined as  $\mathcal{H} = \{R : \|R\| < \infty\}$ . The squared norm for  $(3+1)$ -dimensional space can be defined as [21],

$$\|R\|^2 = \int_{\Sigma_t} \sqrt{-g} g^{tt} R R^* d^3 \Sigma_t. \quad (21)$$

The spatial operator  $A$  is essentially self-adjoint if neither of the solutions of Eq.(20) is square integrable over all space  $L^2(0, \infty)$ . The behavior of the Eq.(20) near  $r \rightarrow 0$  and  $r \rightarrow \infty$  will be considered separately in the following subsections.

Since our aim is to analyse the naked singularity of the KS spacetime, it is important to note that in our analysis, the mass parameter  $M$  and the Hořava parameter  $\omega$  will be chosen in such a way that the inequality  $\omega M^2 < \frac{1}{2}$  holds. Therefore, if  $M = \frac{1}{2}$ , then the Hořava parameter  $\omega < 2$ . In the rest of the paper, the mass parameter and the Hořava parameter are taken as  $M = \frac{1}{2}$  and  $\omega = 1$ , respectively.

#### 1. The case of $r \rightarrow 0$ :

In the case when  $r \rightarrow 0$ , the metric function (6) behave as

$$f(r) \approx 1 - \sqrt{2r} + \mathcal{O}(r^2), \quad (22)$$

thus, the Eq.(20) simplifies to,

$$R'' + \frac{9}{2r} R' - \frac{l(l+1)}{r^2} R = 0, \quad (23)$$

whose solution is

$$R(r) = C_1 r^{\gamma_1} + C_2 r^{\gamma_2}, \quad (24)$$

in which  $C_1, C_2$  are the integration constants and

$$\gamma_1 = \frac{1}{4} \left( -7 + \sqrt{49 + 16l(l+1)} \right), \quad \gamma_2 = -\frac{1}{4} \left( 7 + \sqrt{49 + 16l(l+1)} \right). \quad (25)$$

The square integrability of the solution (24) is checked by calculating the squared norm defined in equation (21) in the limiting case of the metric (5) when  $r \rightarrow 0$ , which is given by

$$\|R\|^2 \sim \int_0^{const.r} \frac{r^2 |R|^2}{(1 - \sqrt{2}r)} dr. \quad (26)$$

We perform the analysis for different modes of solution. If  $l = 0$ , which corresponds to *s-wave* mode, the solution becomes  $R(r) = C_1 + \frac{C_2}{r^{7/2}}$ . The square integrability analysis for this particular solution has revealed that  $\|R\|^2 \rightarrow \infty$ , which is not square integrable, thus, the solution does not belong to the Hilbert space. If  $l \neq 0$ , as long as  $C_1 = 0$  and  $C_2 \neq 0$ , the square integrability condition indicates that  $\|R\|^2 \rightarrow \infty$ , hence the solution does not belong to Hilbert space.

## 2. The case of $r \rightarrow \infty$ :

In the case when  $r \rightarrow \infty$ , the metric function (6) behave as

$$f(r) \approx 1 - \frac{1}{r} + \mathcal{O}(r^{-4}), \quad (27)$$

thus, the Eq.(20) reduces to

$$R'' + \frac{2}{r}R' + (-\tilde{m}^2 \pm i)R = 0, \quad (28)$$

whose solution is given by

$$\psi(r) = \frac{C_3}{r} \sin \kappa r + \frac{C_4}{r} \cos \kappa r, \quad (29)$$

in which  $\kappa = \sqrt{\pm i - \tilde{m}^2}$ , and  $C_3, C_4$  are the integration constants (in general complex). The square integrability is checked with the following norm written for the case  $r \rightarrow \infty$ ,

$$\|R\|^2 \sim \int_{const.}^{\infty} \frac{r^3 |R|^2}{r - 1} dr. \quad (30)$$

The result is that, irrespective of the integration constants, the solution fails to satisfy square integrability condition ( $\|R\|^2 \rightarrow \infty$ ), and hence, does not belong to the Hilbert space.

The method of defining whether the operator  $A$  has a unique self-adjoint extension (or essentially self-adjoint) is to investigate the solution of Eq.(20) in the entire space  $(0, \infty)$  and count the number of solutions that does not belong to the Hilbert space. In other words, if there is one solution that fails to be square integrable for the entire space then the operator  $A$  is said to be essentially self-adjoint. Our analysis has shown that the solutions of the Eq. (20), near  $r \rightarrow 0$  and  $r \rightarrow \infty$ , are not square integrable. Hence, the operator  $A$  is essentially self-adjoint and the future time evolution of the quantum particles/waves can be predicted uniquely. Consequently, the classical naked singularity in the KS spacetime becomes quantum mechanically regular when probed with massive bosons described by the Klein-Gordon equation.

## B. The Dirac fields

The Newman-Penrose formalism will be used to find the Dirac fields propagating in the background geometry of the naked singular KS spacetime. We follow the formalism of Chandrasekhar [27] and, hence, we shift the signature of the metric (5) to  $-2$ . The Chandrasekhar-Dirac (CD) equations in Newman-Penrose formalism are given by

$$\begin{aligned} (D + \epsilon - \rho) F_1 + (\bar{\delta} + \pi - \alpha) F_2 &= 0, \\ (\Delta + \mu - \gamma) F_2 + (\delta + \beta - \tau) F_1 &= 0, \\ (D + \bar{\epsilon} - \bar{\rho}) G_2 - (\delta + \bar{\pi} - \bar{\alpha}) G_1 &= 0, \\ (\Delta + \bar{\mu} - \bar{\gamma}) G_1 - (\bar{\delta} + \bar{\beta} - \bar{\tau}) G_2 &= 0, \end{aligned} \quad (31)$$

where  $F_1, F_2, G_1$  and  $G_2$  are the components of the wave function,  $\epsilon, \rho, \pi, \alpha, \mu, \gamma, \beta$  and  $\tau$  are the spin coefficients. The non-zero spin coefficients are given in Eq.(10). The solution procedure of the set of CD equations (31) is exactly the same as in the references [28, 29]. Thus, applying the same procedures, we end up with a resulting one-dimensional Schrödinger-like wave equation with effective potentials that governs the Dirac field,

$$\left(\frac{d^2}{dr_*^2} + k^2\right) Z_{\pm} = V_{\pm} Z_{\pm}, \quad (32)$$

$$V_{\pm} = \left[ \frac{f\lambda^2}{r^2} \pm \lambda \frac{d}{dr_*} \left( \frac{\sqrt{f}}{r} \right) \right]. \quad (33)$$

In analogy with equation (14), the radial operator  $A$  for the Dirac equations can be written as,

$$A = -\frac{d^2}{dr_*^2} + V_{\pm}.$$

If we write the above operator in terms of the usual coordinates  $r$ , by using  $\frac{d}{dr_*} = f \frac{d}{dr}$ , we have

$$A = -\frac{d^2}{dr^2} - \frac{f'}{f} \frac{d}{dr} + \frac{1}{f^2} \left[ \frac{f\lambda^2}{r^2} \pm \lambda f \frac{d}{dr} \left( \frac{\sqrt{f}}{r} \right) \right]. \quad (34)$$

Our aim now is to investigate whether this radial part of the Dirac operator is essentially self-adjoint or not. We do this by considering Eq.(19) and counting the number of solutions that do not belong to Hilbert space. Thus, Eq.(19) becomes

$$\left( \frac{d^2}{dr^2} + \frac{f'}{f} \frac{d}{dr} - \frac{1}{f^2} \left[ \frac{f\lambda^2}{r^2} \pm \lambda f \frac{d}{dr} \left( \frac{\sqrt{f}}{r} \right) \right] \mp i \right) \psi(r) = 0. \quad (35)$$

The solutions of (35) should be tested for square integrability over all space  $L^2(0, \infty)$ . To do this, the behavior of (35), near  $r \rightarrow 0$  and  $r \rightarrow \infty$  will be considered separately in the following subsections.

#### 1. The case of $r \rightarrow 0$ :

Note that when  $r \rightarrow 0$ , the metric function transforms to (22) and using (22) in equation (35) yields,

$$\psi'' + \frac{1}{2r} \psi' + \frac{\sigma}{r^{3/2}} \psi = 0 \quad (36)$$

where  $\sigma = \frac{\lambda(\lambda \pm 3/2)}{\sqrt{2}}$ , whose solution is

$$\psi(r) = C_5 r^{1/4} J_1 \left( ar^{1/4} \right) + C_6 r^{1/4} N_1 \left( ar^{1/4} \right). \quad (37)$$

in which  $C_5, C_6$  are integration constants and  $a = 4\sqrt{\sigma}$ . The square integrability is checked by using the definition of norm given in Eq.(21), in the limiting case of the metric (5) when  $r \rightarrow 0$ . The result of our analysis is that the solution when  $r \rightarrow 0$  is square integrable, because  $\|\psi\|^2 < \infty$ , indicating that the solution (37) belongs to the Hilbert space.

#### C. The case of $r \rightarrow \infty$ :

In the limiting case of  $r \rightarrow \infty$ , using the metric function (27) in (35), gives

$$\psi'' + \pm i \psi = 0, \quad (38)$$

and its solution is given by,

$$R(r) = C_7 \sin \eta r + C_8 \cos \eta r, \quad (39)$$

in which  $\eta = \frac{1}{\sqrt{2}}(i \pm 1)$ , and  $C_7, C_8$  are the integration constants (in general complex). The square integrability is checked with the norm defined in Eq.(21) written for the case  $r \rightarrow \infty$ . The result is that the solution fails to satisfy square integrability condition ( $\|R\|^2 \rightarrow \infty$ ), and hence, does not belong to the Hilbert space.

In view of the analysis, there is one solution (near,  $r \rightarrow \infty$ ) that do not belong to the Hilbert space in the entire space. As a result, the spatial operator  $A$ , has a unique extension and it is said to be essentially self-adjoint. And, the future time evolution of the Dirac field can be predicted uniquely. Therefore, the naked singularity of the KS spacetime remains quantum regular when probed with fermions ( $spin = 1/2$ ) obeying the CD equations.

#### IV. CONCLUSION AND DISCUSSION

We have studied the KS naked singularity in Hořava's gravity in view of quantum mechanics. In our analysis, we have used the HM criterion that incorporates with the essential self-adjointness of the spatial part of the wave operator  $A$  in the natural Hilbert space of quantum mechanics. This space is a linear function space with square-integrable functions  $L^2(0, \infty)$ .

In our analysis, the KS naked singularity is probed with two different types of quantum fields. First, bosonic waves (scalar wave, with  $spin = 0$ ) governed by the Klein-Gordon equation is used. The calculations have revealed that when the singularity is probed with bosonic waves, the spatial part of the wave operator  $A$  on the KS naked singular spacetime is essentially self-adjoint. Secondly, fermionic waves (Dirac fields,  $spin = 1/2$ ), obeying the CD equation is used. It is shown that, the spatial part of the wave operator on the KS naked singular background, as in the case of bosonic waves, is essentially self-adjoint.

The essential self-adjointness in both probe implies that if quantum field dynamics, in other words, waves are considered in place of classical particles dynamics, i.e. geodesics, the KS naked singularity is "smoothed-out". As a result, the classically KS naked singular spacetimes becomes quantum mechanically wave regular.

The notable outcome of this study is that the quantum nature of the KS naked singularity has a distinctive character when compared with its analog models in classical general relativity. As it was demonstrated in [21, 22] that the naked singularities in negative mass Schwarzschild ( $m < 0$ ), and the extremal Reissner-Nordström ( $|e| > m$ ) spacetimes were quantum mechanically singular.

In a parallel study, the quantum nature of a quantum cosmological model within the context of HL gravity is considered in [30]. It was shown that the quantum Friedmann-Lemaître-Robertson-Walker universe filled with radiation in the context of HL gravity is quantum mechanically nonsingular. The result in cosmological models and our findings for the KS naked singular spacetimes shows that it is possible to heal the apparent singularities in HL gravity within the framework of quantum mechanics.

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- [1] G. T. Horowitz, "Spacetime in string theory", *New J. Phys.* **7**, 201, (2005).
  - [2] M. Natsuume, "The singularity problem in string theory", gr-qc/0108059.
  - [3] A. Ashtekar, "Singularity resolution in loop quantum cosmology: a brief overview", *J. Phys. Conf. Ser.*, **189**, 012003 (2009).
  - [4] P. Hořava, "Membranes at quantum criticality", *J. High Energy Phys.* **03**, 020, (2009).
  - [5] P. Hořava, "Quantum gravity at a Lifshitz point", *Phys. Rev. D* **79**, 084008 (2009).
  - [6] A. Wang, "Hořava Gravity at a Lifshitz Point: A Progress Report", arXiv:1701.06087, To appear in IJMPD.
  - [7] A. Kehagias and K. Sfetsos, "The black hole and FRW geometries of non-relativistic gravity", *Phys. Lett. B* **678**, 123-126 (2009).
  - [8] M. I. Park, "The black hole and cosmological solutions in IR modified Hořava gravity", *J. High Energy Phys.* **9**, 123, (2009).
  - [9] A. Hakimov, B. Turimov, A. Abdujabbarov and B. Ahmedov, "Quantum interference effects in Hořava - Lifshitz gravity", *Mod. Phys. Lett. A* **25**, 3115-3127, (2010).
  - [10] A. Abdujabbarov, B. Ahmedov and A. Hakimov, "A particle motion around black hole in Hořava - Lifshitz gravity", *Phys. Rev. D* **83**, 044053 (2011).
  - [11] A. N. Aliev and Ç. Şentürk, "Slowly rotating black hole solutions to Hořava - Lifshitz gravity", *Phys. Rev. D* **82**, 104016 (2010).



- [12] L. Iorio and M.L. Ruggiero, "Hořava - Lifshitz gravity: Tighter constraints for the Kehagias-Sfetsos solution from new solar system data ", *Int. Jour. of Mod. Phys. D*, **20**, 1079-1093, (2011).
- [13] V. Enolskii, B. Hartmann, V. Kagramanova, J. Kunz, C. Lammerzahl and P. Sirimachan, " Particle motion in Hořava - Lifshitz black hole space-times ", *Phys. Rev. D* **84**, 084011 (2011).
- [14] J. Chen and Y. Wang, " The timelike geodesic motion in Hořava - Lifshitz space-times ", *Int. Jour. of Mod Phys. A*, **25**, 1439 (2010).
- [15] B. Gwak and B.-H. Lee, " Particle probe of Hořava - Lifshitz gravity ", *Jour. Cosmology Astropart. Phys.*1009, 031, (2010).
- [16] L. Iorio and M.L. Ruggiero, " Phenomenological constraints on the Kehagias-Sfetsos solution in the Hořava - Lifshitz gravity from solar system orbital motions ", *Int. Jour. of Mod. Phys. A*, **25**, 5399-5408, (2010).
- [17] M. Liu, J. Lu, B. Yu and J. Lu, " Solar system constraints on asymptotically flat IR modified Hořava gravity through light deflection ", *Gen. Rel. and Grav.*, **43**, 1401-1415, (2011).
- [18] Z. Stuchlik and J. Schee, " Optical effects related to Keplerian discs orbiting Kehagias-Sfetsos naked singularities ", *Class. Quantum Grav.*, **31**, 195014, (2014).
- [19] R. S. S. Vieira, J. Schee, W. Kluzniak, Z. Stuchlik and M. Abramowicz, " Circular geodesics of naked singularities in the Kehagias-Sfetsos metric of Hořava's gravity ", *Phys. Rev. D* **90**, 024035 (2013).
- [20] R. M. Wald, "Dynamics in nonglobally hyperbolic, static sapce-times", *J. Math. Phys. (N.Y.)* **21**, 2082 (1980).
- [21] G. T. Horowitz and D. Marolf, "Quantum probes of spacetime singularities", *Phys. Rev. D* **52**, 5670 (1995).
- [22] A. Ishibashi and A. Hosoya, "Who's afraid of naked singularities ? Probing timelike singularities with finite energy waves", *Phys. Rev. D* **60**, 104028 (1999).
- [23] T. M. Helliwell, D. A. Konkowski and V. Arndt, " Quantum singularity in quasiregular spacetime, as indicated by Klein-Gordon, Maxwell and Dirac fields ", *Gen. Rel. and Grav.* **35**, 79, (2003).
- [24] J. P. M. Pitelli and P. S. Letelier, " Quantum singularities in static spacetimes", *Int. Jour. of Mod. Phys. D*, **20**, 729-743, (2011).
- [25] H. Weyl, "Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen", *Math. Ann.*, **68**, 220-269, (1910).
- [26] J. von Neumann, "Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren", *Math. Ann.*, **102**, 49-131, (1929).
- [27] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, (Oxford University Press, 1992).
- [28] O. Gurtug and T. Tahamtan, " Quantum singularities in a model of  $f(R)$  gravity ", *Eur. Phys. J. C* **72**, 2091 (2012).
- [29] O. Gurtug, M. Halilsoy and S. Habib Mazharimousavi, " Quantum probes of timelike naked singularities in the weak field regime of  $f(R)$  global monopole spacetime ", *J. High Energy Phys.* **01**, 178, (2014).
- [30] J. P. M. Pitelli and A. Saa, " Quantum singularities in Hořava - Lifshitz cosmology ", *Phys. Rev. D* **86**, 063506 (2012).